Mathematics 1 AESB1110: Answers for Exam 1

September 26, 2014

1. Given the points $A(1,1,3)$, $B(3,1,1)$, and $C(1,3,1)$:

   (a) Find the angle $\angle ABC$ $\boxed{\text{total 1 p.}}$

   Answer: Create two vectors $\underline{BA}$ and $\underline{BC}$:

$$\begin{aligned}\underline{BA} &= \langle 1-3, 1-1, 3-1\rangle = \langle -2, 0, 2\rangle\\ \underline{BC} &= \langle 1-3, 3-1, 1-1\rangle = \langle -2, 2, 0\rangle\end{aligned}$$

(1/2)

Use the dot-product formula to compute the cosine of the angle:

$$\begin{aligned}\cos\varphi &= \frac{\underline{BA}\cdot\underline{BC}}{|\underline{BA}|\,|\underline{BC}|},\\ \underline{BA}\cdot\underline{BC} &= -2(-2)+0(2)+2(0) = 4,\\ |\underline{BA}| &= \sqrt{(-2)^2+0^2+2^2} = \sqrt{8} = 2\sqrt{2},\\ |\underline{BC}| &= \sqrt{(-2)^2+2^2+0^2} = \sqrt{8} = 2\sqrt{2},\\ \cos\varphi &= \frac{4}{2\sqrt{2}(2\sqrt{2})} = \frac{1}{2},\\ \varphi &= \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}\ \text{ (or } 60^\circ)\end{aligned}$$

(1/2)

Leaving $\sqrt{8}$ instead of $2\sqrt{2}$ is OK.

(b) Find the area of the triangle ABC $\boxed{\text{total 1 p.}}$

Answer: Use the length of the cross product of the vectors $\underline{BA}$ and $\underline{BC}$:

$$\begin{aligned}\underline{BA} \times \underline{BC} &= \langle -2, 0, 2\rangle \times \langle -2, 2, 0\rangle \\ &= \langle 0(0) - 2(2), -2(2) - (-2)0, -2(2) - 0(-2)\rangle \\ &= \langle -4, -4, -4\rangle \quad \text{(or the opposite vector)}\end{aligned}$$

(1/2)

$$\begin{aligned}|\underline{BA} \times \underline{BC}| &= \sqrt{(-4)^2 + (-4)^2 + (-4)^2} = \sqrt{3(16)} = 4\sqrt{3} \\ S_{\triangle ABC} &= \frac{1}{2}|\underline{BA} \times \underline{BC}| = 2\sqrt{3}\end{aligned}$$

(1/2)

Not simplifying $\sqrt{48}$ as $4\sqrt{3}$ is OK. Geometrical construction instead of the cross-product formula is worth 0.5 p.

(c) Give the equation of the plane passing through these points $\boxed{\text{total 1 p.}}$

Answer: Equation of the plane passing through point $P(x_0, y_0, z_0)$ and having the normal vector $\underline{n} = \langle a, b, c\rangle$ is:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

(1/2)

Take one of the points, say $A(1, 1, 3)$, and $\underline{n} = \underline{BA} \times \underline{BC}$. Then, the equation of the plane is

$$-4(x - 1) - 4(y - 1) - 4(z - 3) = 0.$$

(1/2)

The negative of this equation, or a different reference point are all OK.

(d) Consider the plane passing through the point A and having the normal vector $\underline{AB}$. Find the angle between this plane and the plane constructed in (c)? $\boxed{\text{total 1 p.}}$

Answer: The angle between the planes is the angle between their normal vectors. (1/2)

The plane constructed in (c) has the normal vector $\underline{BA} \times \underline{BC}$. This vector is orthogonal to both $\underline{BC}$ and $\underline{BA}$ (and $\underline{AB} = -\underline{BA}$ as well). Hence, the planes are orthogonal (the angle between them is $\pi/2$). Also OK, if the angle was computed via the dot-product formula. (1/2)

2. Simplify : $\boxed{\text{total 1 p.}}$

$$f(x) = \sin\left(\cos^{-1}(x)\right)$$

Answer: Let $\cos^{-1}(x) = y$. Then, $x = \cos y$. (1/2)
We are looking for $\sin y$. But $\sin y = \sqrt{1-\cos^2 y} = \sqrt{1-x^2}$. Hence,

$$\sin(\cos^{-1}(x)) = \sqrt{1-x^2}$$

(1/2)

Geometrical construction instead of this algebraic one is worth 1/2 p only.

3. Given the function: $\boxed{\text{total 1 p.}}$

$$g(x) = \begin{cases} x^3 - 1, & x < 2, \\ a, & x = 2, \\ \frac{x^2+3x-10}{x-2}, & x > 2. \end{cases}$$

find the value of the constant a for which $g(x)$ is continuous on $\mathbb{R}$.

Answer: A function $g(x)$ is continuous at a iff

$$\lim_{x \to a} g(x) = g(a)$$

The branches of the function are polynomials and therefore continuous on their domains. It remains to analyze the point $x = 2$. Computing the value and the left and right limits of $g(x)$ at $x = 2$ we get (1/2)

$$\begin{aligned} g(2) &= a, \\ \lim_{x \to 2^-} g(x) &= \lim_{x \to 2^{-2}} (x^3 - 1) = 2^3 - 1 = 7 \\ \lim_{x \to 2^+} g(x) &= \lim_{x \to 2^+} \frac{x^2+3x-10}{x-2} = \lim_{x \to 2^+} \frac{(x-2)(x+5)}{x-2} \\ &= \lim_{x \to 2^+} (x+5) = 2 + 5 = 7 \end{aligned}$$

Hence, for the continuity of $g(x)$ at $x = 2$ and thus on $\mathbb{R}$ to hold we need to choose $a = 7$. (1/2)

4. Evaluate the following two limits:

$$\lim_{x\to 0} x^4 \sin\left(\frac{1}{x}\right), \quad \text{and} \quad \lim_{x\to 1} \frac{\sin(1-x)}{x-1}$$

$\boxed{\text{total 2 p.}}$

Answer: For the first limit we use the Squeeze Theorem:

$$\begin{aligned}
&-1 \le \sin\left(\frac{1}{x}\right) \le 1\\
&-x^4 \le x^4 \sin\left(\frac{1}{x}\right) \le x^4\\
&\lim_{x\to 0}(-x^4) = 0 = \lim_{x\to 0} x^4\\
&\lim_{x\to 0} x^4 \sin\left(\frac{1}{x}\right) = 0
\end{aligned}$$

(1)

For the second limit we use the l'Hospital rule:

$$\lim_{x\to 1} \frac{\sin(1-x)}{x-1} = \lim_{x\to 1} \frac{-\cos(1-x)}{1} = -\cos 0 = -1$$

(1)

5. (a) Find y' from the following implicit definition of $y(x)$: $\boxed{\text{total 1 p.}}$

$$e^y \cos(x) = 1 + \sin(xy)$$

Answer:

$$\begin{aligned}
&(e^y \cos(x))' = (1 + \sin(xy))'\\
&(e^y)' \cos(x) + e^y (\cos(x))' = (1)' + (\sin(xy))'\\
&y' e^y \cos(x) - e^y \sin(x) = \cos(xy)(xy)'\\
&y' e^y \cos(x) - e^y \sin(x) = (y + xy')\cos(xy)
\end{aligned}$$

(1/2)

$$\begin{aligned}
&y' e^y \cos(x) - xy' \cos(xy) = e^y \sin(x) + y\cos(xy)\\
&y' \left(e^y \cos(x) - x\cos(xy)\right) = e^y \sin(x) + y\cos(xy)\\
&y' = \frac{e^y \sin(x) + y\cos(xy)}{e^y \cos(x) - x\cos(xy)}
\end{aligned}$$

(1/2)

(b) Prove that $\boxed{\text{total 1 p.}}$

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

Answer: Using implicit differentiation on the equivalent function $\sin(y) = x$: (1/2)

$$\begin{aligned}
&(\sin(y))' = x' \\
&y' \cos(y) = 1 \\
&y' = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}
\end{aligned}$$

(1/2)